

On conjugations of circle homeomorphisms with two break points ¹

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Abstract

Let $f_i \in C^{2+\alpha}(S^1 \setminus \{a_i, b_i\})$, $\alpha > 0$, $i = 1, 2$, be circle homeomorphisms with two break points a_i, b_i i.e. discontinuities in the derivative Df_i , with identical irrational rotation number ρ and $\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, where μ_i are the invariant measures of f_i , $i = 1, 2$. Suppose, the products of the jump ratios of Df_1 and Df_2 do not coincide, i.e. $\frac{Df_1(a_1-0)}{Df_1(a_1+0)} \cdot \frac{Df_1(b_1-0)}{Df_1(b_1+0)} \neq \frac{Df_2(a_2-0)}{Df_2(a_2+0)} \cdot \frac{Df_2(b_2-0)}{Df_2(b_2+0)}$. Then the map ψ conjugating f_1 and f_2 is a singular function, i.e. it is continuous on S^1 , but $D\psi(x) = 0$ a.e. with respect to Lebesgue measure.

1 Introduction

Let f be an orientation preserving homeomorphism of the circle $S^1 \equiv \mathbb{R}/\mathbb{Z}$ with lift $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$, \hat{f} continuous, strictly increasing and $\hat{f}(t+1) = \hat{f}(t) + 1$, $t \in \mathbb{R}$. The circle homeomorphism f is then defined by $f(x) = \hat{f}(\hat{x}) \pmod{1}$, $x \in S^1$, and $x \equiv \hat{x} + \mathbb{Z}$ with $\hat{x} \in [0, 1)$. In the sequel S^1 will be identified with $[0, 1)$ and $x \in S^1$ with $\hat{x} \in [0, 1)$. The interval $[x, y] \subset S^1$ then corresponds to the interval $[\hat{x}, \hat{y}] \subset [0, 2)$. If f is a circle diffeomorphism with irrational rotation number $\rho = \rho_f$ and $\log D\hat{f}$ is of bounded variation, then f is conjugate to the pure rotation f_ρ , that is, there exists an essentially unique homeomorphism φ of the circle with $f = \varphi^{-1} \circ f_\rho \circ \varphi$. This classical result of Denjoy [2] can be extended to circle homeomorphisms with break points. The exact statement of the corresponding theorem will be given later.

It is well known, that circle homeomorphisms f with irrational rotation number ρ_f admit a unique f -invariant probability measure μ_f . Since the conjugating map φ and the invariant measure μ_f are related by $\varphi(x) = \mu_f([0, x])$ (see [8]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f . This problem of smoothness of the conjugacy of smooth diffeomorphisms is by now very well understood (see for instance [1, 15, 9, 10, 11, 19]).

An important class of circle homeomorphisms are homeomorphisms with break points or shortly, class P-homeomorphisms. In general their ergodic properties like the invariant measures, their renormalizations and also their rigidity properties are rather different from those of diffeomorphisms (see [16] chapter I and IV, [9] chapter VI, [13]).

The class of **P-homeomorphisms** consists of orientation preserving circle homeomorphisms f whose lifts \hat{f} are differentiable away from countable many points $\hat{b} \in BP(\hat{f}) \subset$

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$[0, 1)$, corresponding to the so called break points $b \in BP(f) \subset S^1$ of f , at which left and right derivatives, denoted respectively by $D\hat{f}_-$ and $D\hat{f}_+$, exist, such that

- i) there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < D\hat{f}(\hat{x}) < c_2$ for all $\hat{x} \in [0, 1) \setminus BP(\hat{f})$,
 $c_1 < D\hat{f}_-(\hat{b}) < c_2$ and $c_1 < D\hat{f}_+(\hat{b}) < c_2$ for all $\hat{b} \in BP(\hat{f})$,
- ii) $\log D\hat{f}$ has bounded variation in $[0, 1]$.

The ratio $\sigma_f(b) := \frac{D\hat{f}_-(\hat{b})}{D\hat{f}_+(\hat{b})}$ is called the **jump ratio** of f in $b \in BP(f)$. Denote by v the total variation $v = \text{Var}_{[0,1]}(\log D\hat{f})$ of $\log D\hat{f}$ on $[0, 1]$

General P -homeomorphisms with one break point were first studied by K. Khanin and E. Vul in [12]. Among other results it was proved by these authors that their renormalizations approximate fractional linear transformations. Piecewise linear (PL) orientation preserving circle homeomorphisms with break points are the simplest examples in the class of P -homeomorphisms. They show up in many other areas of mathematics as for instance in group theory, homotopy theory and in logic via the Thompson group and its generalizations (see [17]). The invariant measures of PL homeomorphisms were first studied by M. Herman in [9], those of general P -homeomorphisms with one break point by A. Dzhalilov and K. Khanin in [4]. Their main result is the following

Theorem 1.1. *Let f be a P -homeomorphism with one break point b . If the rotation number ρ_f is irrational and $f \in C^{2+\varepsilon}(S^1 \setminus \{b\})$ for some $\varepsilon > 0$, then the f -invariant probability measure μ_f is singular with respect to Lebesgue measure μ_L on S^1 , i.e. there exists a measurable subset $A \subset S^1$ such that $\mu_f(A) = 1$ and $\mu_L(A) = 0$.*

I. Liousse got in [14] the same result for "generic" PL circle homeomorphisms with several break points whose rotation number is irrational and of bounded type. In a next step A. Dzhalilov and I. Liousse [5] and A. Dzhalilov, I. Liousse and D. Mayer [6] studied another class of circle homeomorphisms with two break points. Their main result in [6] is

Theorem 1.2. *Let f be a P -homeomorphism satisfying the following conditions:*

- (a) *the rotation number $\rho = \rho_f$ of f is irrational;*
- (b) *f has two break points b_1, b_2 with $\sigma_f(b_1) \cdot \sigma_f(b_2) \neq 1$;*
- (c) *$D\hat{f}$ is absolutely continuous on every connected interval of $[0, 1] \setminus \{\hat{b}_1, \hat{b}_2\}$ and $D^2\hat{f} \in L^1([0, 1], d\mu_L)$.*

Then the f -invariant probability measure μ_f is singular with respect to Lebesgue measure μ_L .

In the sequel we refer to the smoothness condition (c) in Theorem 1.2 on f as the Katznelson-Ornstein (KO) condition.

The above theorems show that for a sufficiently piecewise smooth circle homeomorphism f with irrational rotation number and one or two break points the map conjugating f and f_ρ is singular. Consider next the regularity properties of the conjugating map between two class P -homeomorphisms with one or two break points and coinciding irrational rotation numbers. The case of one break point with coinciding jump ratios, the so called rigidity problem, was studied in great detail by K. Khanin and D. Khmelev in [13] and by A. Teplinskii and K. Khanin in [18].

If $\rho = [k_1, k_2, \dots, k_n, \dots]$ is the continued fraction expansion of the irrational rotation number ρ , define the sets

$$M_o = \{\rho : \forall n \in \mathbb{N} \exists C > 0 : k_{2n-1} \leq C\},$$

$$M_e = \{\rho : \forall n \in \mathbb{N} \exists C > 0 : k_{2n} \leq C\}.$$

The main result of [18] is then the following

Theorem 1.3. (*Teplinskii-Khanin*). *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\})$, $i = 1, 2$, be P -homeomorphisms each with one break point b_i . Assume*

- (1) *their rotation numbers $\rho(f_i)$, $i = 1, 2$, are irrational and coincide, i.e. $\rho(f_1) = \rho(f_2) = \rho$, $\rho \in \mathbb{R}^1 \setminus \mathbb{Q}$;*
- (2) *their jump ratios $\sigma_i = \sigma_{f_i}(b_i)$, $i = 1, 2$, coincide, i.e. $\sigma_1 = \sigma_2 = \sigma$.*

Then the map ψ conjugating the homeomorphisms f_1 and f_2 is a C^1 -diffeomorphism of the circle if either $\sigma > 1$ and $\rho \in M_o$ or $\sigma < 1$ and $\rho \in M_e$.

In the case of not coinciding jump ratios A. Dzhaliylov, H. Akin and S. Temir [7] proved

Theorem 1.4. *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\})$, $i = 1, 2$, be P -homeomorphisms each with one break point b_i . Assume*

- (1) *their rotation numbers ρ_i , $i = 1, 2$, are irrational and coincide i.e. $\rho_1 = \rho_2 = \rho$, $\rho \in \mathbb{R}^1 \setminus \mathbb{Q}$;*
- (2) *their jump ratios $\sigma_{f_i}(b_i)$, $i = 1, 2$, are positive and do not coincide.*

Then the homeomorphism ψ conjugating f_1 and f_2 is a singular function, i.e. ψ is continuous on S^1 and $D\psi(x) = 0$ a.e. with respect to Lebesgue measure μ_L .

In the present paper we will extend this result to circle homeomorphisms with coinciding irrational rotation numbers having each two break points. Our main result is the following

Theorem 1.5. *Let $f_i \in C^{2+\alpha}(S^1 \setminus \{a_i, b_i\})$, $i = 1, 2$, be P -homeomorphisms each with two break points a_i, b_i . Assume*

- (1) *their rotation numbers $\rho(f_i)$, $i = 1, 2$, are irrational and coincide i.e. $\rho(f_1) = \rho(f_2) = \rho$, $\rho \in \mathbb{R}^1 \setminus \mathbb{Q}$;*
- (2) *the products of their jump ratios $\sigma_{f_i}(a_i) \cdot \sigma_{f_i}(b_i)$ do not coincide i.e. $\sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \neq \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)$;*
- (3) *$\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, where μ_i is the invariant probability measure of f_i , $i = 1, 2$.*

Then the map ψ conjugating f_1 and f_2 is singular.

2 Preliminaries and Notations

Consider an orientation preserving circle homeomorphism f with lift \hat{f} and irrational rotation number $\rho = \rho_f$. If the rotation number ρ has the continued fraction expansion $\rho = [k_1, k_2, \dots, k_n, \dots] = 1/(k_1 + 1/(k_2 + \dots + 1/(k_n + \dots)))$ its convergents p_n/q_n , $n \in \mathbb{N}$, are defined by $p_n/q_n = [k_1, k_2, \dots, k_n]$. Then the denominators q_n satisfy the well known recursion relation $q_{n+1} = k_{n+1}q_n + q_{n-1}$, $n \geq 1$, $q_0 = 1$, $q_1 = k_1$.

For an arbitrary point $x_0 \in S^1$ define $\Delta_0^{(n)}(x_0)$ as the closed interval in S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$, such that for n odd x_{q_n} is to the left of x_0 and for n even it is to its right with respect to the orientation induced from the real line. Denote by $\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0))$, $i \geq 1$, the iterates of the interval $\Delta_0^{(n)}(x_0)$ under f . It is well known, that the set $\xi_n(x_0)$ of intervals with mutually disjoint interiors defined as

$$\xi_n(x_0) = \left\{ \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n \right\} \cup \left\{ \Delta_j^{(n)}(x_0), 0 \leq j < q_{n-1} \right\} \quad (1)$$

determines a partition of the circle for any n . The partition $\xi_n(x_0)$ is called the n -th **dynamical partition** of the point x_0 with **generators** $\Delta_0^{(n-1)}(x_0)$ and $\Delta_0^{(n)}(x_0)$. Obviously, the partition $\xi_{n+1}(x_0)$ is a refinement of the partition $\xi_n(x_0)$: indeed the intervals of order n belong to $\xi_{n+1}(x_0)$ and each interval $\Delta_i^{(n-1)}(x_0) \in \xi_n(x_0)$, $0 \leq i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\xi_{n+1}(x_0)$ such that

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

Recall the following definition introduced in [10]:

Definition 2.1. *An interval $I = (x, y) \subset S^1$ is q_n -small and its endpoints x, y are q_n -close if the intervals $f^i(I)$, $0 \leq i < q_n$, are disjoint.*

It is clear that the interval (x, y) is q_n -small if, depending on the parity of n , either $y \prec x \preceq f^{q_{n-1}}(y) \prec y$ or $f^{q_{n-1}}(x) \preceq y \prec x \prec f^{q_{n-1}}(x)$ in the order induced from the real line.

Then we can show

Lemma 2.2. *Let f be a P -homeomorphism with a finite number of break points b_i , $i = 1, 2, \dots, m$, and irrational rotation number ρ . Assume $x, y \in S^1$ are q_n -close and $b_i \notin \{f^j(x), f^j(y), 0 \leq j < q_n\}$, $i = 1, 2, \dots, m$. Then for any $0 \leq k < q_n$ the following inequality holds:*

$$e^{-v} \leq \frac{D\hat{f}^k(\hat{x})}{D\hat{f}^k(\hat{y})} \leq e^v. \quad (2)$$

where v is the total variation of $\log D\hat{f}$ on $[0, 1]$ and \hat{x}, \hat{y} are the lifts of x, y to the interval $[0, 1)$.

Proof. Take any two q_n -close points $x, y \in S^1$ and $0 \leq k < q_n$. Denote by I the open interval with endpoints x and y . Because the intervals $f^s(I)$, $0 \leq s < k$ are disjoint, we obtain

$$|\log D\hat{f}^k(\hat{x}) - \log D\hat{f}^k(\hat{y})| \leq \sum_{s=0}^{k-1} |\log D\hat{f}(\hat{f}^s(\hat{x})) - \log D\hat{f}(\hat{f}^s(\hat{y}))| \leq v,$$

from which inequality (2) follows immediately. \square

The following Lemma can be proven easily using Lemma 2.2.

Lemma 2.3. *Let f be a P -homeomorphism with a finite number of break points b_i , $i = 1, 2, \dots, m$, and irrational rotation number ρ . If $x_0 \in S^1$, $n \geq 1$ and $b_i \notin \{f^j(x_0), 0 \leq j < q_n\}$ for $i = 1, 2, \dots, m$, then*

$$e^{-v} \leq \prod_{i=0}^{q_n-1} D\hat{f}(\hat{f}^i(\hat{x}_0)) \leq e^v. \quad (3)$$

Inequality (3) is called the **Denjoy inequality**. The proof of Lemma 2.3 is as for circle diffeomorphisms (see for instance [11]). Using Lemma 2.3 it can be shown that the intervals of the dynamical partition $\xi_n(x_0)$ in (1) have exponentially small length. Indeed one finds

Corollary 2.4. *Let $\Delta^{(n)}$ be an arbitrary element of the dynamical partition $\xi_n(x_0)$. Then*

$$l(\Delta^{(n)}) := \mu_L(\Delta^{(n)}) \leq \text{const } \lambda^n, \quad (4)$$

where $\lambda = (1 + e^{-v})^{-1/2} < 1$.

From Corollary 2.4 it follows that the trajectory of every point $x \in S^1$ is dense in S^1 . This together with monotonicity of the homeomorphism f implies the following

Theorem 2.5. *Suppose that a homeomorphism f satisfies the conditions of Lemma 2.3. Then f is topologically conjugate to the linear rotation f_ρ .*

In the following discussion we have to compare different intervals. For this we use

Definition 2.6. Let $C > 1$. We call two intervals of S^1 **C-comparable** if the ratio of their lengths is in $[C^{-1}, C]$.

Corollary 2.7. *Suppose the homeomorphism f satisfies the conditions of Lemma 2.3. Then for any interval $I \subset S^1$ the intervals I and $f^{q_n}(I)$ are e^v -comparable. If the interval I is q_n -small then $l(f^i(I)) < \text{const } \lambda^n$ for all $i = 0, 1, \dots, q_n - 1$.*

3 The Cross-ratio Tools

Let us first recall two definitions:

Definition 3.1. The **cross-ratio** $Cr(a_1, a_2, a_3, a_4)$ of four strictly ordered points $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$, is defined as

$$Cr(a_1, a_2, a_3, a_4) = \frac{(a_2 - a_1)(a_4 - a_3)}{(a_3 - a_1)(a_4 - a_2)}.$$

Definition 3.2. The **cross-ratio distortion** $Dist(a_1, a_2, a_3, a_4; f)$ of four strictly ordered points $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4$ with respect to a strictly increasing function f on \mathbb{R} is defined as

$$Dist(a_1, a_2, a_3, a_4; f) = \frac{Cr(f(a_1), f(a_2), f(a_3), f(a_4))}{Cr(a_1, a_2, a_3, a_4)}.$$

For $k \geq 3$ let $\hat{z}_i \in [a, a+1] \subset \mathbb{R}$, $i = 1, \dots, k$ be the lifts of the points $z_i \in S^1$, $i = 1, \dots, k$, with $z_1 \prec z_2 \prec \dots \prec z_k \prec z_1$ such that $\hat{z}_1 < \hat{z}_2 < \dots < \hat{z}_k$. The vector $(\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k) \in \mathbb{R}^k$ is called the **lifted vector** of $(z_1, z_2, \dots, z_k) \in (S^1)^k$. Consider a circle homeomorphism f with lift \hat{f} . We define the cross-ratio distortion of (z_1, z_2, z_3, z_4) with respect to f by $Dist(z_1, z_2, z_3, z_4; f) := Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; \hat{f})$ where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4) . We need the following

Lemma 3.3. (see [4]) *Suppose f is a P -homeomorphism with a finite number of break points and $f \in C^{2+\alpha}(S^1 \setminus BP(f))$ for some $\alpha > 0$. Consider any four points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and $[z_1, z_4] \subset S^1 \setminus BP(f)$. Then*

$$|Dist(z_1, z_2, z_3, z_4; f) - 1| \leq K|\hat{z}_4 - \hat{z}_1|^{1+\alpha}$$

for some positive constant K depending only on f .

Next we consider the case when the interval $[z_1, z_4]$ contains one break point b of the homeomorphism f . We estimate the distortion of its cross-ratio when b lies outside the middle interval $[z_2, z_3]$. For this we define for $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ and $b \in [z_1, z_2] \cup [z_3, z_4]$ the following quantities:

$$\alpha := \hat{z}_2 - \hat{z}_1, \beta := \hat{z}_3 - \hat{z}_2, \gamma := \hat{z}_4 - \hat{z}_3, \tau := \hat{z}_2 - \hat{b}, \xi := \frac{\beta}{\alpha}, \zeta := \frac{\tau}{\alpha}, \eta := \frac{\beta}{\gamma}, \vartheta := \frac{\hat{b} - \hat{z}_3}{\gamma}.$$

Lemma 3.4. *Assume f is P -homeomorphism with a finite number of break points and $f \in C^2(S^1 \setminus BP(f))$. Choose points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ such that f has one single break point b in $[z_1, z_2] \cup [z_3, z_4]$. Then*

$$i) |Dist(z_1, z_2, z_3, z_4; f) - \frac{[\sigma_f(b) + (1 - \sigma_f(b))\zeta](1 + \xi)}{\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi}| \leq K_1|\hat{z}_4 - \hat{z}_1|, \text{ if } b \in [z_1, z_2],$$

$$ii) |Dist(z_1, z_2, z_3, z_4; f) - \frac{[\sigma_f(b) + (1 - \sigma_f(b))\vartheta](1 + \eta)}{\sigma_f(b) + (1 - \sigma_f(b))\vartheta + \eta}| \leq K_1|\hat{z}_4 - \hat{z}_1|, \text{ if } b \in [z_3, z_4]$$

for some positive constant K_1 depending only on f .

Proof. We prove only the first assertion of Lemma 3.4. The second one can be proved similarly. Obviously

$$\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1) = D\hat{f}_+(\hat{b})(\hat{z}_2 - \hat{b}) + \frac{D^2\hat{f}(\theta_1)(\hat{z}_2 - \hat{b})^2}{2} + D\hat{f}_-(\hat{b})(\hat{b} - \hat{z}_1) + \frac{D^2\hat{f}(\theta_2)(\hat{b} - \hat{z}_1)^2}{2}$$

and

$$\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1) = D\hat{f}_+(\hat{b})(\hat{z}_3 - \hat{b}) + \frac{D^2\hat{f}(\theta_3)(\hat{z}_3 - \hat{b})^2}{2} + D\hat{f}_-(\hat{b})(\hat{b} - \hat{z}_1) + \frac{D^2\hat{f}(\theta_4)(\hat{b} - \hat{z}_1)^2}{2},$$

for some $\theta_1 \in (\hat{b}, \hat{z}_2)$, $\theta_2 \in (\hat{z}_1, \hat{b})$, $\theta_3 \in (\hat{b}, \hat{z}_3)$, $\theta_4 \in (\hat{z}_1, \hat{b})$.

Using the last two relations it is easy to show

$$\frac{\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1)}{\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1)} = \frac{\sigma_f(b) + (1 - \sigma_f(b))\zeta + O(\alpha)}{G(\zeta, \xi) + O(\alpha + \beta)} \times \frac{\alpha}{\alpha + \beta} \quad (5)$$

where $G(\zeta, \xi) = (\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi)/(1 + \xi)$ and $\xi > 0$. It is clear that $\min\{1, \sigma_f(b)\} \leq \sigma_f(b) + (1 - \sigma_f(b))\zeta \leq \max\{1, \sigma_f(b)\}$ and $\min\{1, \sigma_f(b)\} \leq G(\zeta, \xi) \leq 1 + \max\{1, \sigma_f(b)\}$. The last two inequalities together with (5) imply that

$$\frac{\hat{f}(\hat{z}_2) - \hat{f}(\hat{z}_1)}{\hat{f}(\hat{z}_3) - \hat{f}(\hat{z}_1)} : \frac{\alpha}{\alpha + \beta} = \frac{[\sigma_f(b) + (1 - \sigma_f(b))\zeta](1 + \xi)}{\sigma_f(b) + (1 - \sigma_f(b))\zeta + \xi} + O(\alpha + \beta). \quad (6)$$

Since $\hat{f} \in C^2([\hat{z}_2, \hat{z}_4])$, we get

$$\frac{\hat{f}(\hat{z}_4) - \hat{f}(\hat{z}_3)}{\hat{f}(\hat{z}_4) - \hat{f}(\hat{z}_2)} : \frac{\gamma}{\gamma + \beta} = 1 + O(\gamma + \beta). \quad (7)$$

The relations (6) and (7) imply the first assertion of Lemma 3.4. The second one can be proved similarly. \square

4 Proof of Theorem 1.5.

For the proof of Theorem 1.5 we need several Lemmas which we formulate next. Their proofs will be given later. Consider two copies of the circle S^1 and homeomorphisms f_i each with two break points a_i, b_i , $i = 1, 2$, and the same irrational rotation number ρ . Assume that f_1 and f_2 satisfy the conditions of Theorem 1.5.

Let φ_1 and φ_2 be maps conjugating f_1 and f_2 with the pure rotation f_ρ , i.e. $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$ and $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$. It is easy to check that the map $\psi = \varphi_2^{-1} \circ \varphi_1$ conjugates f_1 and f_2 , i.e.

$$\psi(f_1(x)) = f_2(\psi(x)) \quad (8)$$

for all $x \in S^1$. By assumption in Theorem 1.5 $\sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \neq \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)$. W.l.o.g assume $\sigma_{f_1}(a_1) \neq \sigma_{f_2}(a_2)$. Since φ_i , $i = 1, 2$, is unique up to an additive constant we can choose φ_i , $i = 1, 2$, such that $\varphi_1(a_1) = a_1$ and $\varphi_2^{-1}(a_1) = a_2$ and hence $\psi(a_1) = a_2$. Then by assumption of Theorem 1.5 $\psi(b_1) = b_2$. Recall, that the length of an interval $[a, b] \subset S^1$ is defined by

$$l([a, b]) := \mu_L([a, b]) = \begin{cases} \hat{b} - \hat{a}, & \text{if } 0 \leq \hat{a} < \hat{b} < 1, \\ 1 + \hat{b} - \hat{a}, & \text{if } 0 \leq \hat{b} < \hat{a} < 1. \end{cases}$$

Definition 4.1. Let $R_1 > 1$ and $\varepsilon > 0$ be constants. The points $x_0, z_i \in S^1$ with $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ satisfy **conditions** $(C_{R_1, \varepsilon})$ if:

- (a) $R_1^{-1}l([z_2, z_3])\sqrt{\varepsilon} \leq l([z_1, z_2]) \leq R_1l([z_2, z_3])\sqrt[4]{\varepsilon}$;
- (b) $R_1^{-1}l([z_2, z_3]) \leq l([z_3, z_4]) \leq R_1l([z_2, z_3])$;
- (c) $\max_{1 \leq i \leq 4} l([x_0, z_i]) \leq R_1l([z_2, z_3])$.

For $x_0 \in S^1$ with lift \hat{x}_0 in $[0, 1)$ define $d(\hat{x}_0) := \min\{\hat{x}_0, (1 - \hat{x}_0)\}$.

Lemma 4.2. Assume, that the lift $\hat{\psi}$ of the conjugating map ψ has a positive derivative $D\hat{\psi}(\hat{x}_0) = \omega$ at the point $\hat{x}_0 \in [0, 1)$ and let $R_1 > 1$ be a constant. Then there exists a constant $C_2 = C_2(\omega, R_1)$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\hat{x}_0, \varepsilon) \in (0, d(\hat{x}_0))$ such that for all $z_i \in S^1$ with $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$, $i = 1, 2, 3, 4$, satisfying the conditions $(C_{R_1, \varepsilon})$ one has:

- i) $\frac{l([z_1, z_2])}{l([z_2, z_3])}(1 - C_2\sqrt{\varepsilon}) \leq \frac{l[\psi(z_1), \psi(z_2)]}{l[\psi(z_2), \psi(z_3)]} \leq \frac{l([z_1, z_2])}{l([z_2, z_3])}(1 + C_2\sqrt{\varepsilon})$,
- ii) $\frac{l([z_3, z_4])}{l([z_2, z_3])}(1 - C_2\varepsilon) \leq \frac{l[\psi(z_3), \psi(z_4)]}{l[\psi(z_2), \psi(z_3)]} \leq \frac{l([z_3, z_4])}{l([z_2, z_3])}(1 + C_2\varepsilon)$.

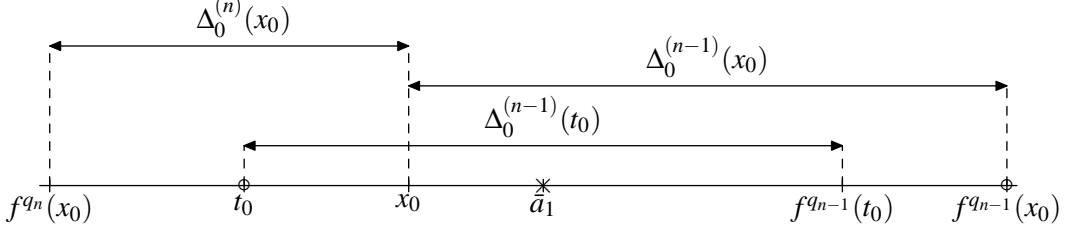


Figure 1: The point $\bar{a}_1 = f_1^{-l}(a_1)$ belongs to the interval $[f_1^{q_n}(x_0), f_1^{q_{n-1}}(x_0)]$ and is the middle point of $[t_0, f_1^{q_{n-1}}(t_0)]$.

Lemma 4.3. *Suppose the lift $\hat{\psi}$ has a positive derivative $D\hat{\psi}(\hat{x}_0) = \omega$ at the point $\hat{x}_0 \in [0, 1)$ and let $R_1 > 1$ be a constant. Then there exists a constant $R_2 = R_2(\omega, R_1)$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\hat{x}_0, \varepsilon) \in (0, d(\hat{x}_0))$ such that for all $z_i \in S^1$ with $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$, $i = 1, 2, 3, 4$, satisfying the conditions $(C_{R_1, \varepsilon})$ one has:*

$$|Dist(z_1, z_2, z_3, z_4; \psi) - 1| \leq R_2 \sqrt{\varepsilon}. \quad (9)$$

The main idea for proving that the map ψ conjugating f_1 and f_2 is a singular function is to construct a quadruple of points z_i , $i = 1, 2, 3, 4$, for which the ratio of the distortions $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $Dist(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})$ stays away from 1.

For this assume $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for the lift $\hat{x}_0 \in [0, 1)$ of a point $x_0 \in S^1$. W.l.o.g. we can choose n to be odd. Then we have $\Delta_0^{(n)}(z) = [f_1^{q_n}(z), z]$ and $\Delta_0^{(n-1)}(z) = [z, f_1^{q_{n-1}}(z)]$ for any point z of the circle. Consider the n -th dynamical partition $\xi_n(x_0)$ of the point $x_0 \in S^1$ defined by the homeomorphism f_1 . Only one interval of the partition $\xi_n(x_0)$ covers the break point a_1 . Hence there exists a unique point \bar{a}_1 with either $\bar{a}_1 \in \Delta_0^{(n-1)}(x_0)$ and $f_1^l(\bar{a}_1) = a_1$ for some $0 \leq l < q_n$, or $\bar{a}_1 \in \Delta_0^{(n)}(x_0)$ and $f_1^l(\bar{a}_1) = a_1$ for some $0 \leq l < q_{n-1}$. We call the point \bar{a}_1 the q_n -**preimage** of the break point a_1 in $\Delta_0^{(n-1)}(x_0) \cup \Delta_0^{(n)}(x_0)$. There exists a unique point t_0 such that \bar{a}_1 is the **middle point** of the interval $[t_0, f_1^{q_{n-1}}(t_0)]$ (see Figure 1). Consider now the n -th dynamical partitions $\xi_n(t_0)$ of the point t_0 defined by f_1 on the first circle respectively $\zeta_n(\psi(t_0))$ of the point $\psi(t_0)$ defined by f_2 on the second circle. For each $n \geq 1$ define

$$\Delta_i^{(n)}(t_0) := f_1^i(\Delta_0^{(n)}(t_0)), \quad C_i^{(n)}(\psi(t_0)) := f_2^i(C_0^{(n)}(\psi(t_0))), \quad 0 \leq i < q_{n+1},$$

where $\Delta_0^{(n)}(t_0)$ respectively $C_0^{(n)}(\psi(t_0))$ are the initial intervals of order n of the points t_0 respectively $\psi(t_0)$ determined by f_1 respectively f_2 . By definition

$$\xi_n(t_0) = \{\Delta_i^{(n-1)}(t_0), 0 \leq i < q_n\} \cup \{\Delta_j^{(n)}(t_0), 0 \leq j < q_{n-1}\},$$

$$\zeta_n(\psi(t_0)) = \{C_i^{(n-1)}(\psi(t_0)), 0 \leq i < q_n\} \cup \{C_j^{(n)}(\psi(t_0)), 0 \leq j < q_{n-1}\}.$$

Since the common rotation number ρ of f_1 and f_2 is irrational, the order of the points on the orbit $\{f_1^k(x_0), k \in \mathbb{Z}^1\}$ on the first circle will be precisely the same as the one for the orbit $\{f_2^k(\psi(x_0)), k \in \mathbb{Z}^1\}$ on the second circle. This together with the relation $\psi(f_1(x)) = f_2(\psi(x))$ for $x \in S^1$ implies that

$$\psi(\Delta_i^{(n-1)}(t_0)) = C_i^{(n-1)}(\psi(t_0)), \quad 0 \leq i < q_n, \quad \psi(\Delta_j^{(n)}(t_0)) = C_j^{(n)}(\psi(t_0)), \quad 0 \leq j < q_{n-1}.$$

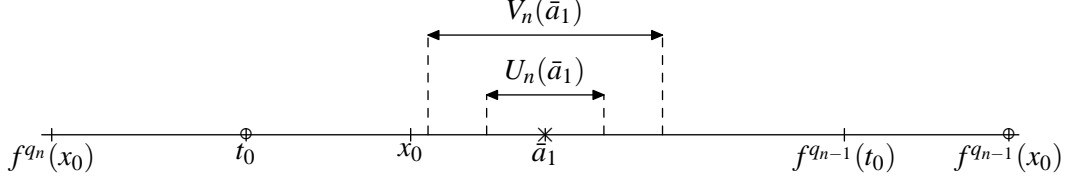


Figure 2: The intervals $U_n(\bar{a}_1)$ and $V_n(\bar{a}_1)$ are $\sqrt{\varepsilon}$ and $\sqrt[4]{\varepsilon}$ comparable with $[t_0, f_1^{q_{n-1}}(t_0)]$ respectively.

Denote by \bar{b}_1 the q_n -preimage of the second break point b_1 of f_1 in $\Delta_0^{(n-1)}(t_0) \cup \Delta_0^{(n)}(t_0)$, such that $f_1^p(\bar{b}_1) = b_1$ for some $0 \leq p < q_n$. The ψ -preimages of the points \bar{a}_1 and \bar{b}_1 lie in $C_0^{(n-1)}(\psi(t_0)) \cup C_0^{(n)}(\psi(t_0))$. Using relation (5) we get

$$f_2^l(\psi(\bar{a}_1)) = f_2^{l-1}(f_2(\psi(\bar{a}_1))) = f_2^{l-1}(\psi(f_1(\bar{a}_1))) = \dots = \psi(f_1^l(\bar{a}_1)) = \psi(a_1) = a_2.$$

Similarly one shows $f_2^p(\psi(\bar{b}_1)) = b_2$.

For $\varepsilon > 0$ define the two neighbourhoods U_n, V_n of the point $\bar{a}_1 \in S^1$ as

$$U_n(\bar{a}_1) = \{z \in S^1 : \hat{z} \in (\hat{a}_1 - \delta_n, \hat{a}_1 + \delta_n) \text{ with } \delta_n = \frac{1}{4}l([t_0, f_1^{q_{n-1}}(t_0)])\sqrt{\varepsilon}\},$$

$$V_n(\bar{a}_1) = \{z \in S^1 : \hat{z} \in (\hat{a}_1 - \gamma_n, \hat{a}_1 + \gamma_n) \text{ with } \gamma_n = \frac{1}{2}l([t_0, f_1^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}\}.$$

It is clear that $U_n(\bar{a}_1) \subset V_n(\bar{a}_1) \subset [t_0, f_1^{q_{n-1}}(t_0)]$ (see Figure 2).

Then two cases are possible:

either $\bar{b}_1 \in U_n(\bar{a}_1)$ **or** $\bar{b}_1 \notin U_n(\bar{a}_1)$ i.e. $\bar{b}_1 \in [f^{q_n}(t_0), f^{q_{n-1}}(t_0)] \setminus U_n(\bar{a}_1)$.

Consider first the case $\bar{b}_1 \in U_n(\bar{a}_1)$. If \hat{b}_1 lies on the left hand side of the point \hat{a}_1 we define

$$\begin{aligned} \hat{z}_1 &:= \hat{a}_1 - \frac{1}{2}l([t_0, f^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}, \quad \hat{z}_2 := \hat{a}_1, \\ \hat{z}_3 &:= \hat{a}_1 + \frac{1}{4}l([t_0, f^{q_{n-1}}(t_0)]), \quad \hat{z}_4 := \hat{a}_1 + \frac{1}{2}l([t_0, f^{q_{n-1}}(t_0)]), \end{aligned} \quad (10)$$

corresponding to the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_2 = \bar{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$.

If on the other hand \hat{b}_1 is on right hand side of \hat{a}_1 , we define

$$\begin{aligned} \hat{z}_1 &:= \hat{a}_1 - \frac{1}{2}l([t_0, f^{q_{n-1}}(t_0)]), \quad \hat{z}_2 := \hat{a}_1 - \frac{1}{4}l([t_0, f^{q_{n-1}}(t_0)]), \\ \hat{z}_3 &:= \hat{a}_1, \quad \hat{z}_4 := \hat{a}_1 + \frac{1}{2}l([t_0, f^{q_{n-1}}(t_0)])\sqrt[4]{\varepsilon}, \end{aligned} \quad (11)$$

corresponding to the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with $z_3 = \bar{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$.

in the following we consider only the first case, the second one can be handled similarly.

Then one shows

Lemma 4.4. *Suppose the circle homeomorphism f_1 satisfies the conditions of Lemma 2.3. Let $\delta > 0$ be the constant determined by Lemma 4.2 and let for large enough n the points \hat{z}_i , $i = 1, 2, 3, 4$ defined in (10) be the lifts of the points $z_i \in S^1$, $i = 1, 2, 3, 4$. Then the triple of intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$ has the following properties:*

- (1) $[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset U_\delta(x_0) = \{z \in S^1 : \hat{x} \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)\};$

(2) the intervals $[z_s, z_{s+1}]$, $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$, $s = 1, 2, 3$, satisfy conditions $(C_{R_1, \varepsilon})$ for some constant $R_1 > 1$ depending only on the variation v of $\log Df_1$.

Lemma 4.5. Assume the circle homeomorphisms f_i , $i = 1, 2$, satisfy the conditions of Theorem 1.5. Let $z_i \in S^1$, $i = 1, 2, 3, 4$, be the points defined in Lemma 4.4. Then the following inequalities hold for sufficiently large n :

$$\left| \text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n}) - \sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \right| \leq R_2 \sqrt[4]{\varepsilon}, \quad (12)$$

$$\left| \text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n}) - \sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2) \right| \leq R_2 \sqrt[4]{\varepsilon}, \quad (13)$$

where the positive constant $R_2 = R_2(R_1, f_1, f_2)$ does not depend on ε .

After these preparations we can now proceed to the proof of Theorem 1.5.

Proof of Theorem 1.5. Let f_1 and f_2 be circle homeomorphisms satisfying the conditions of Theorem 1.5. The lift $\hat{\psi}(\hat{x})$ of the conjugating map $\psi(x)$ is a continuous and monotone increasing function on R^1 . Hence $\hat{\psi}(\hat{x})$ has a finite derivative $D\hat{\psi}(\hat{x})$ almost everywhere (w.r.t. Lebesgue measure) on R^1 . Recall that $D\hat{\psi}(\hat{x} + 1) = D\hat{\psi}(\hat{x})$ for each $\hat{x} \in R^1$ where the derivative $D\hat{\psi}(\hat{x})$ is defined. It is enough to show that $D\hat{\psi}(\hat{x}) = 0$ for almost all points \hat{x} of the interval $[0, 1)$. Suppose $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for some point $\hat{x}_0 \in [0, 1)$ corresponding to the point $x_0 \in S^1$. Choose an $\varepsilon > 0$ and the points $z_i \in S^1$, $i = 1, 2, 3, 4$, with lifted vector $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ as defined in (10). Then by the second assertion of Lemma 4.4 the intervals $[z_s, z_{s+1}]$, $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$, $s = 1, 2, 3$, satisfy conditions $(C_{R_1, \varepsilon})$ for some constant $R_1 > 1$ depending only on the variation v of $\log Df_1$.

Lemma 4.3 next implies

$$|\text{Dist}(z_1, z_2, z_3, z_4; \psi) - 1| \leq R_2 \sqrt{\varepsilon} \quad (14)$$

and

$$|\text{Dist}(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi) - 1| \leq R_2 \sqrt{\varepsilon}. \quad (15)$$

Hence

$$\left| \frac{\text{Dist}(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi)}{\text{Dist}(z_1, z_2, z_3, z_4; \psi)} - 1 \right| \leq R_3 \sqrt{\varepsilon}, \quad (16)$$

where the constant $R_3 > 0$ does not depend on ε and n .

But by definition

$$\begin{aligned} \text{Dist}(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi) &= \\ &= \frac{Cr(\psi(f_1^{q_n}(z_1)), \psi(f_1^{q_n}(z_2)), \psi(f_1^{q_n}(z_3)), \psi(f_1^{q_n}(z_4)))}{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))}. \end{aligned}$$

Since ψ is conjugating f_1 and f_2 we can readily see that

$$\begin{aligned} Cr(\psi(f_1^{q_n}(z_1)), \psi(f_1^{q_n}(z_2)), \psi(f_1^{q_n}(z_3)), \psi(f_1^{q_n}(z_4))) &= \\ &= Cr(f_2^{q_n}(\psi(z_1)), f_2^{q_n}(\psi(z_2)), f_2^{q_n}(\psi(z_3)), f_2^{q_n}(\psi(z_4))). \end{aligned}$$

It now follows that

$$\begin{aligned}
& \frac{Dist(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi)}{Dist(z_1, z_2, z_3, z_4; \psi)} = \\
& = \frac{Cr(\psi(f_1^{q_n}(z_1)), \psi(f_1^{q_n}(z_2)), \psi(f_1^{q_n}(z_3)), \psi(f_1^{q_n}(z_4)))}{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))} \times \frac{Cr(z_1, z_2, z_3, z_4)}{Cr(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4))} = \\
& = \frac{Cr(f_2^{q_n}(\psi(z_1)), f_2^{q_n}(\psi(z_2)), f_2^{q_n}(\psi(z_3)), f_2^{q_n}(\psi(z_4)))}{Cr(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4))} : \frac{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))}{Cr(z_1, z_2, z_3, z_4)} = \\
& = \frac{Dist(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})}{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}.
\end{aligned}$$

Combining this with inequality (16) we get

$$\left| \frac{Dist(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})}{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})} - 1 \right| \leq R_3 \sqrt{\varepsilon}. \quad (17)$$

But using Lemma 4.5 we get

$$\left| \frac{Dist(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})}{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})} - \frac{\sigma_{f_2}(a_2) \cdot \sigma_{f_2}(b_2)}{\sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1)} \right| \leq Const \sqrt[4]{\varepsilon} \quad (18)$$

for sufficiently large n . This contradiction proves Theorem 1.5 in the first case.

There remains the case where the point \bar{b}_1 belongs to the set $[f_1^{q_n}(t_0), f_1^{q_{n-1}}(t_0)] \setminus U_n(\bar{a}_1)$.

Let $\hat{\hat{b}}_1$ lie on the left hand side of the point $\hat{\hat{a}}_1$, the case $\hat{\hat{b}}_1$ on the right hand side of $\hat{\hat{a}}_1$ can be handled similarly. We define

$$\begin{aligned}
\hat{z}_1 &:= \hat{\hat{a}}_1 - \frac{1}{4}l([t_0, f^{q_{n-1}}(t_0)])\sqrt{\varepsilon}, \quad \hat{z}_2 := \hat{\hat{a}}_1, \\
\hat{z}_3 &:= \hat{\hat{a}}_1 + \frac{1}{4}l([t_0, f^{q_{n-1}}(t_0)]), \quad \hat{z}_4 := \hat{\hat{a}}_1 + \frac{1}{2}l([t_0, f^{q_{n-1}}(t_0)]),
\end{aligned} \quad (19)$$

which determine the points $z_i \in S^1$, $i = 1, 2, 3, 4$ with $z_2 = \bar{a}_1$ and $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. The proof of Theorem 1.5 for the corresponding intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$, proceeds now exactly as in the previous case. This concludes the proof of Theorem 1.5.

5 Proofs of Lemmas 4.2 - 4.5.

We start with the proof of Lemma 4.2.

Proof. Suppose, the derivative $D\hat{\psi}(\hat{x}_0)$ exists and $D\hat{\psi}(\hat{x}_0) = \omega > 0$ for the lift $\hat{x}_0 \in [0, 1)$ of some point x_0 in S^1 . By the definition of the derivative there exists for any $\varepsilon > 0$ a number $\delta = \delta(x_0, \varepsilon) \in (0, d(x_0))$ such that, for all $\hat{x} \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta)$,

$$\omega - \varepsilon < \frac{\hat{\psi}(\hat{x}) - \hat{\psi}(\hat{x}_0)}{\hat{x} - \hat{x}_0} < \omega + \varepsilon. \quad (20)$$

Now take four points $\hat{z}_i \in (\hat{x}_0 - \delta, \hat{x}_0 + \delta) \subset [0, 1]$ satisfying conditions $(C_{R_1, \varepsilon})$. W.l.o.g. we can assume that $[z_1, z_4] \subset U_\delta(x_0)$ and $z_1 \prec z_4 \prec x_0 \prec z_1$. Relation (20) then implies for $\hat{x} = \hat{z}_i$, $i = 1, 2, 3, 4$

$$(\omega - \varepsilon)(\hat{x}_0 - \hat{z}_i) < \hat{\psi}(\hat{x}_0) - \hat{\psi}(\hat{z}_i) < (\omega + \varepsilon)(\hat{x}_0 - \hat{z}_i).$$

This yields the following inequalities for \hat{z}_s , $s = 1, 2, 3$

$$\begin{aligned} \omega - \varepsilon \frac{(\hat{x}_0 - \hat{z}_{s+1}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s} &< \frac{\hat{\psi}(\hat{z}_{s+1}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s} \\ &< \omega + \varepsilon \frac{(\hat{x}_0 - \hat{z}_{s+1}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s}, \end{aligned} \quad (21)$$

respectively for $s = 1, 2$

$$\begin{aligned} \omega - \varepsilon \frac{(\hat{x}_0 - \hat{z}_{s+2}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s} &\leq \frac{\hat{\psi}(\hat{z}_{s+2}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s} \\ &\leq \omega + \varepsilon \frac{(\hat{x}_0 - \hat{z}_{s+2}) + (\hat{x}_0 - \hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s}. \end{aligned} \quad (22)$$

Since the points z_i , $i = 1, 2, 3, 4$, satisfy conditions $(C_{R_1, \varepsilon})$, it is easy to show that

$$\max_{1 \leq i \leq 4} \left\{ \frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_2 - \hat{z}_1} \right\} \leq R_1 \frac{\hat{z}_3 - \hat{z}_2}{\hat{z}_2 - \hat{z}_1} \leq \frac{R_1^2}{\sqrt{\varepsilon}}, \quad (23)$$

$$\max_{1 \leq i \leq 4} \left\{ \frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_3 - \hat{z}_2}, \frac{\hat{x}_0 - \hat{z}_i}{\hat{z}_4 - \hat{z}_3} \right\} \leq R_1^2. \quad (24)$$

Combining relations (21), (22), (23) and (24) we get

$$\omega - C_4 \sqrt{\varepsilon} \leq \frac{\hat{\psi}(\hat{z}_2) - \hat{\psi}(\hat{z}_1)}{\hat{z}_2 - \hat{z}_1} \leq \omega + C_4 \sqrt{\varepsilon}; \quad (25)$$

for $l = 2, 3$ we get

$$\omega - C_4 \varepsilon \leq \frac{\hat{\psi}(\hat{z}_{l+1}) - \hat{\psi}(\hat{z}_l)}{\hat{z}_{l+1} - \hat{z}_l} \leq \omega + C_4 \varepsilon, \quad (26)$$

respectively for $s = 1, 2$

$$\omega - C_4 \varepsilon \leq \frac{\hat{\psi}(\hat{z}_{s+2}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+2} - \hat{z}_s} \leq \omega + C_4 \varepsilon, \quad (27)$$

where the constant $C_4 > 0$ depends on R_1 , ω , but does not depend on $l([\hat{z}_s, \hat{z}_{s+1}])$, $s = 1, 2, 3$, and on ε . Using the equality

$$\frac{\hat{\psi}(\hat{z}_{s+1}) - \hat{\psi}(\hat{z}_s)}{\hat{\psi}(\hat{z}_s) - \hat{\psi}(\hat{z}_{s-1})} \cdot \frac{\hat{z}_{s+1} - \hat{z}_s}{\hat{z}_s - \hat{z}_{s-1}} = \frac{\hat{\psi}(\hat{z}_{s+1}) - \hat{\psi}(\hat{z}_s)}{\hat{z}_{s+1} - \hat{z}_s} \cdot \frac{\hat{z}_s - \hat{z}_{s-1}}{\hat{\psi}(\hat{z}_s) - \hat{\psi}(\hat{z}_{s-1})}$$

and relations (25), (26), (27) we get the assertions of Lemma 4.2. \square

Next we will prove Lemma 4.3.

Proof. Since

$$\begin{aligned} \text{Dist}(z_1, z_2, z_3, z_4, \psi) &= \frac{Cr(\hat{\psi}(\hat{z}_1), \hat{\psi}(\hat{z}_2), \hat{\psi}(\hat{z}_3), \hat{\psi}(\hat{z}_4))}{Cr(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)} = \\ &= \frac{\hat{\psi}(\hat{z}_2) - \hat{\psi}(\hat{z}_1)}{\hat{z}_2 - \hat{z}_1} \cdot \frac{\hat{\psi}(\hat{z}_4) - \hat{\psi}(\hat{z}_3)}{\hat{z}_4 - \hat{z}_3} \cdot \frac{\hat{z}_3 - \hat{z}_1}{\hat{\psi}(\hat{z}_3) - \hat{\psi}(\hat{z}_1)} \cdot \frac{\hat{z}_4 - \hat{z}_2}{\hat{\psi}(\hat{z}_4) - \hat{\psi}(\hat{z}_2)} \end{aligned}$$

inequalities (25)-(27) prove Lemma 4.3. \square

We continue with the proof of Lemma 4.4.

Proof. We assume n to be odd. Hence $f_1^{q_n}(z) \prec z \prec f_1^{q_{n-1}}(z) \prec f_1^{q_n}(z)$ for any point z on the circle S^1 . The point \bar{a}_1 lies in the interval $[f_1^{q_n}(x_0), f_1^{q_{n-1}}(x_0)]$ and is the middle point of the interval $[t_0, f_1^{q_{n-1}}(t_0)]$. This and the structure of the orbits imply $x_{-3q_{n-1}} \prec f_1^{q_n}(t_0) \prec t_0 \prec f_1^{q_{n-1}}(t_0) \prec x_{3q_{n-1}}$. By construction $[z_1, z_4] \subset [t_0, f_1^{q_{n-1}}(t_0)]$. Consequently $[f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [f_1^{q_n}(t_0), f_1^{q_n+q_{n-1}}(t_0)]$. Summarizing we get therefore

$$[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [x_{-3q_{n-1}}, x_{3q_{n-1}}]. \quad (28)$$

Obviously

$$\begin{aligned} [x_{-3q_{n-1}}, x_{3q_{n-1}}] &= [x_{-3q_{n-1}}, x_{-2q_{n-1}}] \cup [x_{-2q_{n-1}}, x_{-q_{n-1}}] \cup [x_{-q_{n-1}}, x_0] \cup \\ &\cup [x_0, x_{q_{n-1}}] \cup [x_{q_{n-1}}, x_{2q_{n-1}}] \cup [x_{2q_{n-1}}, x_{3q_{n-1}}]. \end{aligned} \quad (29)$$

By Corollary 2.7 the intervals $[x, y]$, $[f_1^{q_{n-1}}(x), f_1^{q_{n-1}}(y)]$ and $[f_1^{-q_{n-1}}(x), f_1^{-q_{n-1}}(y)]$ are e^{v_1} -comparable for any $x, y \in S^1$. This together with equation (29) and Corollary 2.4 implies that

$$l([x_{-3q_{n-1}}, x_{3q_{n-1}}]) \leq (1 + 5e^{3v_1})l([x_0, x_{q_{n-1}}]) \leq \text{const} \lambda_1^n,$$

for a constant $\lambda_1 \in (0, 1)$. For sufficiently large n then obviously $[x_{-3q_{n-1}}, x_{3q_{n-1}}] \subset (x_0 - \delta, x_0 + \delta)$. This together with (28) implies the first assertion of Lemma 4.4.

Next we will prove the second assertion of Lemma 4.4 .

By Corollary 2.7 the intervals $[z_s, z_{s+1}]$ and $[f^{q_n}(z_s), f^{q_n}(z_{s+1})]$ are e^{v_1} -comparable for all $s = 1, 2, 3$. Using the definition of the points $z_s, s = 1, 2, 3, 4$ and the Denjoy inequality it is easy to verify that these intervals satisfy the assumptions a) and b) of conditions $(C_{R_1, \varepsilon})$. Using the relations

$$[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset [x_{-3q_{n-1}}, x_{3q_{n-1}}]$$

we get

$$\max_{1 \leq s \leq 4} \left\{ |\hat{x}_0 - \hat{z}_s|, |\hat{x}_0 - \hat{y}_s| \right\} \leq l([x_{-3q_{n-1}}, x_{3q_{n-1}}]) \quad (30)$$

where $y_s := f_1^{q_n}(z_s)$ $s = 1, 2, 3, 4$. Now we want to compare the lengths of the intervals $[x_{-3q_{n-1}}, x_{3q_{n-1}}]$ and $[t_0, f_1^{q_{n-1}}(t_0)]$. Using the definition of t_0 it is easy to see that $x_{-2q_{n-1}} \prec t_0 \prec x_{2q_{n-1}}$. Applying $f_1^{sq_{n-1}}, s \in \mathbb{Z}$, to these relations we get $x_{(s-2)q_{n-1}} \prec t_{sq_{n-1}} \prec$

$x_{(s+2)q_{n-1}}, s \in \mathbb{Z}$. In particular the last relations imply $t_{-5q_{n-1}} \prec x_{-3q_{n-1}}, x_{3q_{n-1}} \prec t_{5q_{n-1}}$ and hence $[x_{-3q_{n-1}}, x_{3q_{n-1}}] \subset [t_{-5q_{n-1}}, t_{5q_{n-1}}]$. Consequently

$$l([x_{-3q_{n-1}}, x_{3q_{n-1}}]) \leq l([t_{-5q_{n-1}}, t_{5q_{n-1}}]). \quad (31)$$

But the intervals $[t_{-5q_{n-1}}, t_{5q_{n-1}}]$ and $[t_0, f_1^{q_{n-1}}(t_0)]$ are $(1+2e^{v_1}+2e^{2v_1}+2e^{3v_1}+2e^{4v_1}+e^{5v_1})$ -comparable. This together with eq. (30) implies $l([x_{-3q_{n-1}}, x_{3q_{n-1}}]) \leq 10e^{5v_1}l([t_0, f_1^{q_{n-1}}(t_0)])$. Finally, we conclude that the points $z_s, s = 1, 2, 3, 4$ and $f_1^{q_n}(z_s), s = 1, 2, 3, 4$, satisfy conditions $(C_{R_1, \varepsilon})$ with the constant $R_1 = 40e^{5v_1}$. This concludes the proof of Lemma 4.4. \square

Remains the proof of Lemma 4.5.

Proof. Choose the points $z_s, s = 1, 2, 3, 4$, according to formulas (10) and consider the two sets of intervals $\{f_1^i[z_s, z_{s+1}], 0 \leq i < q_n, s = 1, 2, 3\}$ and $\{f_2^i[\psi(z_s), \psi(z_{s+1})], 0 \leq i < q_n, s = 1, 2, 3\}$. By the construction of the intervals $[z_s, z_{s+1}], s = 1, 2, 3$, only the intervals $f_1^l([z_1, z_2])$ and $f_1^p([z_1, z_2])$ cover the break points a_1 respectively b_1 , namely $a_1 = f_1^l(z_2), b_1 \in f_1^p[z_1, z_2]$.

Similarly, alone the intervals $f_2^l[\psi(z_1), \psi(z_2)]$ respectively $f_2^p[\psi(z_1), \psi(z_2)]$ cover the break points a_2 respectively b_2 , namely $a_2 = f_2^l(\psi(z_2))$ and $b_2 \in f_2^p[\psi(z_1), \psi(z_2)]$. Next we compare the distortions $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $Dist(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})$. We estimate only the first distortion, the second one can be estimated analogously. Rewriting it as

$$\begin{aligned} Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) &= \prod_{\substack{i=0 \\ i \neq l, p}}^{q_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) \times \\ &\times Dist(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1) \times Dist(f_1^p(z_1), f_1^p(z_2), f_1^p(z_3), f_1^p(z_4); f_1), \end{aligned} \quad (32)$$

we apply Lemma 3.3 to obtain

$$\begin{aligned} &\prod_{\substack{i=0 \\ i \neq l, p}}^{q_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) = \\ &= \exp\left\{\sum_{\substack{i=0 \\ i \neq l, p}}^{q_n-1} \log(1 + O(|[f_1^i(z_1), f_1^i(z_4)]|^{1+\alpha}))\right\}. \end{aligned} \quad (33)$$

By construction $[f_1^i(z_1), f_1^i(z_4)] \subset [f_1^i(t_0), f_1^i(f_1^{q_{n-1}}(t_0))]$ for $0 \leq i < q_n$. By Corollary 2.4 the length of the last interval is bounded by $\text{const } \lambda_1^n$. Thus we get for $0 \leq i < q_n$

$$l([f_1^i(z_1), f_1^i(z_4)]) \leq \text{const } \lambda_1^n. \quad (34)$$

The interval $[t_0, f_1^{q_{n-1}}(t_0)]$ is q_n -small and hence

$$\sum_{i=0}^{q_n-1} l([f_1^i(z_1), f_1^i(z_4)]) \leq 1. \quad (35)$$

Combining equations (33), (34), (35) we get

$$\begin{aligned}
& \left| \prod_{\substack{i=0 \\ i \neq l, p}}^{q_n-1} \text{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1 \right| \leq \\
& \leq \text{const } \lambda_1^{n\alpha} \sum_{\substack{i=0 \\ i \neq l, p}}^{q_n-1} l([f_1^i(z_1), f_1^i(z_4)]) \leq \text{const } \lambda_1^{n\alpha}.
\end{aligned} \tag{36}$$

Next we estimate the distortions

$$\text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1) \text{ and } \text{Dist}(f_1^p(z_1), f_1^p(z_2), f_1^p(z_3), f_1^p(z_4); f_1).$$

Define for $0 \leq m < q_n$ the length ratios

$$\vartheta(m) := \frac{l([f_1^m(z_2), f_1^m(z_3)])}{l([f_1^m(z_1), f_1^m(z_2)])}, \quad \tau(m) := \frac{l([f_1^m(\bar{b}_1), f_1^m(z_2)])}{l([f_1^m(z_1), f_1^m(z_2)])}. \tag{37}$$

Lemma 2.2 then implies the following inequalities

$$e^{-v_1} \cdot \vartheta(0) \leq \vartheta(m) \leq e^{v_1} \cdot \vartheta(0), \quad e^{-v_1} \cdot \tau(0) \leq \tau(m) \leq e^{v_1} \cdot \tau(0). \tag{38}$$

Using the definitions of the points $z_i, i = 1, 2, 3$ we get

$$\vartheta(0) = \frac{1}{2\sqrt[4]{\varepsilon}}, \quad 0 \leq \tau(0) \leq \frac{\sqrt[4]{\varepsilon}}{4}. \tag{39}$$

Define next for $x > 0$ and $0 \leq t \leq 1$ the functions $G(x, \sigma)$ and $F(x, t, \sigma)$ as

$$G(x, \sigma) := \frac{\sigma(1+x)}{\sigma+x}, \quad F(x, t, \sigma) := \frac{[\sigma + (1-\sigma)t](1+x)}{\sigma + (1-\sigma)t + x}. \tag{40}$$

Applying Lemma 3.4 we get

$$|\text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1) - G(\vartheta(l), \sigma_{f_1}(a_1))| \leq K_2 l([f_1^l(z_1), f_1^l(z_4)]), \tag{41}$$

$$\begin{aligned}
& |\text{Dist}(f_1^p(z_1), f_1^p(z_2), f_1^p(z_3), f_1^p(z_4); f_1) - F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1))| \leq \\
& K_2 l([f_1^p(z_1), f_1^p(z_4)]).
\end{aligned} \tag{42}$$

The definitions of the functions G, F together with equations (37) and (38) imply

$$\begin{aligned}
& |G(\vartheta(l), \sigma_{f_1}(a_1)) - G(\vartheta(0), \sigma_{f_1}(a_1))| \leq K_3 \sqrt[4]{\varepsilon}, \\
& |F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1)) - F(\vartheta(0), \tau(0), \sigma_{f_1}(b_1))| \leq K_3 \sqrt[4]{\varepsilon},
\end{aligned} \tag{43}$$

$$\begin{aligned}
& |G(\vartheta(0), \sigma_{f_1}(a_1)) - \sigma_{f_1}(a_1)| \leq K_3 \sqrt[4]{\varepsilon}, \\
& |F(\vartheta(0), \tau(0), \sigma_{f_1}(b_1)) - \sigma_{f_1}(b_1)| \leq K_3 \sqrt[4]{\varepsilon},
\end{aligned} \tag{44}$$

where the constant K_3 is given by

$$K_3 = (\sigma_{f_1}(a_1)|1 - \sigma_{f_1}(a_1)| + \sigma_{f_1}(b_1)|1 - \sigma_{f_1}(b_1)|)(1 + e^{v_1}).$$

The last four equations imply

$$|G(\vartheta(l), \sigma_{f_1}(a_1)) - \sigma_{f_1}(a_1)| \leq 2K_3 \sqrt[4]{\varepsilon}, \quad (45)$$

$$|F(\vartheta(p), \tau(p), \sigma_{f_1}(b_1) - \sigma_{f_1}(b_1))| \leq 2K_3 \sqrt[4]{\varepsilon}. \quad (46)$$

Combining equations (31), (32), (35) and (40)-(46) we obtain finally

$$\left| \text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n}) - \sigma_{f_1}(a_1) \cdot \sigma_{f_1}(b_1) \right| \leq R_2 \sqrt[4]{\varepsilon},$$

which proves the first inequality in Lemma 4.5. The second inequality can be proven by using similar arguments as above. \square

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